

TURÁN TYPE INEQUALITIES FOR KRÄTZEL FUNCTIONS

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ABSTRACT. Complete monotonicity, Laguerre and Turán type inequalities are established for the so-called Krätzel function Z_ρ^ν , defined by

$$Z_\rho^\nu(u) = \int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{u}{t}} dt,$$

where $u > 0$ and $\rho, \nu \in \mathbb{R}$. Moreover, we prove the complete monotonicity of a determinant function of which entries involve the Krätzel function.

1. Introduction

In 1941 while studying the zeros of Legendre polynomials, the Hungarian mathematician Paul Turán discovered the following inequality

$$[P_n(x)]^2 > P_{n-1}(x)P_{n+1}(x),$$

where $|x| < 1$, $n \in \{1, 2, \dots\}$ and P_n stands for the classical Legendre polynomial. This inequality was published by P. Turán only in 1950 in [31]. However, since the publication in 1948 by G. Szegő [30] of the above famous Turán inequality for Legendre polynomials, many authors have deduced analogous results for classical (orthogonal) polynomials and special functions. In the last 62 years it has been shown by several researchers that the most important (orthogonal) polynomials (e.g. Laguerre, Hermite, Appell, Bernoulli, Jacobi, Jensen, Pollaczek, Lommel, Askey-Wilson, ultraspherical polynomials) and special functions (e.g. Bessel, modified Bessel, gamma, polygamma, Riemann zeta functions) satisfy a Turán inequality. In 1981 one of the PhD students of P. Turán, L. Alpár [2] in Turán's biography mentioned that the above Turán inequality had a wide-ranging effect, this inequality was dealt with in more than 60 papers. The Turán type inequalities now have a more extensive literature and recently some of the results have been applied successfully in problems that arise in information theory, economic theory and biophysics. Motivated by these applications, the Turán type inequalities have recently come under the spotlight once again and it has been shown that, for example, the Gauss and Kummer hypergeometric functions, as well as the generalized hypergeometric functions, satisfy naturally some Turán type inequalities. For the most recent papers on this subject we refer to [3], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [17], [21], [19], [29]. For more details see also the references therein.

Motivated by the above immense research on Turán type inequalities, in this paper our aim is to deduce complete monotonicity, lower bounds and Turán type inequalities for the so-called Krätzel function, defined below. The Krätzel function is defined for $u > 0$, $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$, being such that $\operatorname{Re}(\nu) < 0$ for $\rho \leq 0$, by the integral

$$(1.1) \quad Z_\rho^\nu(u) = \int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{u}{t}} dt.$$

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For $\rho \geq 1$ the function (1.1) was introduced by E. Krätzel [24] as a kernel of the integral transform

$$(K_\nu^\rho f)(u) = \int_0^\infty Z_\rho^\nu(ut) f(t) dt,$$

which was applied to the solution of some ordinary differential equations. The study of the Krätzel function (1.1) and the above integral transform were continued by several authors. For example, in [23] the authors deduced explicit forms of Z_ρ^ν in terms of the generalized Wright function, while in [22] the authors obtained the asymptotic behavior of this function at zero and infinity and gave applications to evaluation of integrals involving Z_ρ^ν . Such investigations now are of a great interest in connection with applications, see [22] and [23] and the references therein for more details. We note that the Krätzel function occurs in the study of astrophysical thermonuclear functions, which are derived on the basis of Boltzmann-Gibbs statistical mechanics, see [28]. It is also important to note that the Krätzel function Z_1^ν is related to the modified Bessel function of the second kind K_ν . More precisely, in view of the formula [4, p. 237]

$$(1.2) \quad K_\nu(u) = \frac{u^\nu}{2^{\nu+1}} \int_0^\infty t^{-\nu-1} e^{-t-\frac{u^2}{4t}} dt$$

we have that for all $u > 0$ and $\nu \in \mathbb{C}$

$$Z_1^\nu\left(\frac{u^2}{4}\right) = 2\left(\frac{u}{2}\right)^\nu K_{-\nu}(u) = 2\left(\frac{u}{2}\right)^\nu K_\nu(u)$$

and consequently

$$(1.3) \quad Z_1^\nu(u) = 2u^{\nu/2} K_\nu(2\sqrt{u}).$$

Note that this function is useful in chemical physics. More precisely, the function

$$u \mapsto 2^{\nu-1} Z_1^\nu(uq^2/4) = (q\sqrt{u})^\nu K_\nu(q\sqrt{u})$$

is related to the Hartree-Fock energy and is used as a basis function for the helium isoelectronic series. See [15] and [16] for more details. Moreover, the function

$$u \mapsto \sqrt{2/\pi} 2^{\nu-1} Z_1^\nu(u^2/4) = \sqrt{2/\pi} u^\nu K_\nu(u)$$

is called in the literature as reduced Bessel function and plays an important role in theoretical chemistry. See [33] and the references therein for more details.

This paper is organized as follows: in the next section we present some monotonicity, log-convexity properties, complete monotonicity and lower bounds for the Krätzel function, while in the third section we prove the complete monotonicity of a Turán determinant which entries involve the Krätzel function Z_ρ^ν . The main result of the third section is actually a generalization of a Turán-type inequality, of which counterpart is a conjecture at the end of this paper.

We close these preliminaries with the following definitions, which will be used in the sequel. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is said to be completely monotonic if f has derivatives of all orders and satisfies

$$(-1)^m f^{(m)}(u) \geq 0$$

for all $u > 0$ and $m \in \{0, 1, \dots\}$. A function $g : (0, \infty) \rightarrow (0, \infty)$ is said to be logarithmically convex, or simply log-convex, if its natural logarithm $\ln g$ is convex, that is, for all $u_1, u_2 > 0$ and $\alpha \in [0, 1]$ we have

$$g(\alpha u_1 + (1 - \alpha)u_2) \leq [g(u_1)]^\alpha [g(u_2)]^{1-\alpha}.$$

Note that every completely monotonic function is log-convex, see [34, p. 167].

2. Monotonicity and convexity properties of the Krätzel functions

Our first main result reads as follows:

Theorem 1. *If $\nu, \rho \in \mathbb{R}$ and $u > 0$, then the following assertions are true:*

a. *The Krätzel function Z_ρ^ν satisfies the recurrence relation*

$$(2.1) \quad \nu Z_\rho^\nu(u) = \rho Z_\rho^{\nu+\rho}(u) - u Z_\rho^{\nu-1}(u).$$

b. *The function $u \mapsto Z_\rho^\nu(u)$ is completely monotonic on $(0, \infty)$.*

c. *The function $\nu \mapsto Z_\rho^\nu(u)$ is log-convex on \mathbb{R} .*

d. *The function $u \mapsto Z_\rho^\nu(u)$ is log-convex on $(0, \infty)$.*

e. *For $n \in \{1, 2, \dots\}$ the following Laguerre type inequality holds*

$$(2.2) \quad \left[[Z_\rho^\nu(u)]^{(n)} \right]^2 \leq [Z_\rho^\nu(u)]^{(n-1)} [Z_\rho^\nu(u)]^{(n+1)}.$$

f. *Suppose that $\rho > 0$. Then the following inequality holds*

$$(2.3) \quad Z_\rho^{-\nu}(u) \geq 2u^{\frac{1}{2}-\nu} \Gamma(\nu) K_1(2\sqrt{u})$$

provided that $\nu \geq 1$. Moreover, if $0 < \nu \leq 1$, then the above inequality is reversed. In particular, the inequality

$$(2.4) \quad u^{\nu-1} K_\nu(u) \geq 2^{\nu-1} \Gamma(\nu) K_1(u)$$

is valid for all $\nu \geq 1$. If $0 < \nu \leq 1$, then the above inequality is reversed. In the above inequality equality holds if u tends to zero or $\nu = 1$.

We note here that similar result to that of (2.4) was proved by M.E.H. Ismail [18]. More precisely, M.E.H. Ismail proved among others that for all $u > 0$ and $\nu > 1/2$ the inequality

$$(2.5) \quad u^\nu K_\nu(u) e^u > 2^{\nu-1} \Gamma(\nu)$$

is valid and it is sharp as $u \rightarrow 0$. It should be mentioned here that for $\nu \geq 1$ the inequality (2.4) is better than (2.5) since for all $\nu \geq 1$ and $u > 0$ we have

$$u^\nu K_\nu(u) \geq 2^{\nu-1} \Gamma(\nu) u K_1(u) > 2^{\nu-1} \Gamma(\nu) e^{-u}.$$

Observe that in fact (2.5) and the later inequality follow from the fact (see [25]) that the function

$$u \mapsto u^\nu e^u K_\nu(u)$$

is strictly increasing on $(0, \infty)$ for all $\nu > 1/2$ and consequently for all $u > 0$ we have $u e^u K_1(u) > 1$. Here we used tacitly that when $\nu > 0$ is fixed and u tends to zero, the asymptotic relation

$$u^\nu K_\nu(u) \sim 2^{\nu-1} \Gamma(\nu)$$

holds.

Proof of Theorem 1. a. By using (1.3) and the recurrence relation [32, p. 79]

$$K_{\nu-1}(u) - K_{\nu+1}(u) = -\frac{2\nu}{u} K_\nu(u)$$

we obtain

$$u Z_1^{\nu-1}(u) - Z_1^{\nu+1}(u) = -\nu Z_1^\nu(u).$$

We note that the above recurrence relation can be verified also by using integration by parts as follows

$$\begin{aligned} \nu Z_1^\nu(u) &= \int_0^\infty \left(e^{-t-\frac{u}{t}} \right) (\nu t^{\nu-1}) dt \\ &= \int_0^\infty \left(1 - \frac{u}{t^2} \right) \left(e^{-t-\frac{u}{t}} \right) t^\nu dt \\ &= Z_1^{\nu+1}(u) - u Z_1^{\nu-1}(u). \end{aligned}$$

Moreover, by using the same idea we immediately have

$$\begin{aligned}\nu Z_\rho^\nu(u) &= \int_0^\infty \left(e^{-t^\rho - \frac{u}{t}}\right) (\nu t^{\nu-1}) dt \\ &= \int_0^\infty \left(\rho t^{\rho-1} - \frac{u}{t^2}\right) \left(e^{-t^\rho - \frac{u}{t}}\right) t^\nu dt \\ &= \rho Z_\rho^{\nu+\rho}(u) - u Z_\rho^{\nu-1}(u).\end{aligned}$$

b. The change of variable $1/t = s$ in (1.1) yields

$$(2.6) \quad Z_\rho^\nu(u) = \int_0^\infty \left(s^{-\nu-1} e^{-s^{-\rho}}\right) e^{-us} ds,$$

i.e. the Krätzel function Z_ρ^ν is the Laplace transform of the function $s \mapsto s^{-\nu-1} e^{-s^{-\rho}}$. This in view of the Bernstein-Widder theorem (see [34]) implies that the function $u \mapsto Z_\rho^\nu(u)$ is completely monotonic, i.e. for all $n \in \{0, 1, \dots\}$, $\nu, \rho \in \mathbb{R}$ and $u > 0$ we have

$$(-1)^n [Z_\rho^\nu(u)]^{(n)} > 0.$$

We note that this can be verified also directly by using that for all $n \in \{0, 1, \dots\}$, $\nu, \rho \in \mathbb{R}$ and $u > 0$

$$(2.7) \quad [Z_\rho^\nu(u)]^{(n)} = (-1)^n Z_\rho^{\nu-n}(u),$$

which follows via mathematical induction easily from (1.1) or (2.6).

c. Recall the Hölder-Rogers inequality [27, p. 54], that is,

$$(2.8) \quad \int_a^b |f(t)g(t)| dt \leq \left[\int_a^b |f(t)|^p dt \right]^{1/p} \left[\int_a^b |g(t)|^q dt \right]^{1/q},$$

where $p > 1$, $1/p + 1/q = 1$, f and g are real functions defined on $[a, b]$ and $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$. Using (1.1) and (2.8) we obtain that

$$\begin{aligned}Z_\rho^{\alpha\nu_1+(1-\alpha)\nu_2}(u) &= \int_0^\infty t^{\alpha\nu_1+(1-\alpha)\nu_2-1} e^{-t^\rho - \frac{u}{t}} dt \\ &= \int_0^\infty t^{\alpha(\nu_1-1)+(1-\alpha)(\nu_2-1)} e^{-t^\rho - \frac{u}{t}} dt \\ &= \int_0^\infty \left(t^{\nu_1-1} e^{-t^\rho - \frac{u}{t}}\right)^\alpha \left(t^{\nu_2-1} e^{-t^\rho - \frac{u}{t}}\right)^{1-\alpha} dt \\ &\leq \left[\int_0^\infty t^{\nu_1-1} e^{-t^\rho - \frac{u}{t}} dt \right]^\alpha \left[\int_0^\infty t^{\nu_2-1} e^{-t^\rho - \frac{u}{t}} dt \right]^{1-\alpha} \\ &= [Z_\rho^{\nu_1}(u)]^\alpha [Z_\rho^{\nu_2}(u)]^{1-\alpha}\end{aligned}$$

holds for all $\alpha \in [0, 1]$, $\nu_1, \nu_2, \rho \in \mathbb{R}$ and $u > 0$, i.e. the function $\nu \mapsto Z_\rho^\nu(u)$ is log-convex on \mathbb{R} .

d. This follows from the fact that the integrand in (1.1) or (2.6) is a log-linear function of u and by using the Hölder-Rogers inequality (2.8) we have that

$$\begin{aligned}Z_\rho^\nu(\alpha u_1 + (1-\alpha)u_2) &= \int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{1}{t}(\alpha u_1 + (1-\alpha)u_2)} dt \\ &= \int_0^\infty \left(t^{\nu-1} e^{-t^\rho - \frac{u_1}{t}}\right)^\alpha \left(t^{\nu-1} e^{-t^\rho - \frac{u_2}{t}}\right)^{1-\alpha} dt \\ &\leq \left[\int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{u_1}{t}} dt \right]^\alpha \left[\int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{u_2}{t}} dt \right]^{1-\alpha} \\ &= [Z_\rho^\nu(u_1)]^\alpha [Z_\rho^\nu(u_2)]^{1-\alpha}\end{aligned}$$

holds for all $\alpha \in [0, 1]$, $\nu, \rho \in \mathbb{R}$ and $u_1, u_2 > 0$, i.e. the function $u \mapsto Z_\rho^\nu(u)$ is log-convex on $(0, \infty)$.

Alternatively, we may use part **b** of this theorem. More precisely, it is known that every completely monotonic function is log-convex (see [34, p. 167]), and then in view of part **b** the Krätzel function Z_ρ^ν is log-convex on $(0, \infty)$. Moreover, as a third proof we may use part **c** of this theorem. Namely, since the function $\nu \mapsto Z_\rho^\nu(u)$ is log-convex, it follows that the following Turán type inequality holds for all $\nu_1, \nu_2, \rho \in \mathbb{R}$ and $u > 0$

$$(2.9) \quad \left[Z_\rho^{\frac{\nu_1 + \nu_2}{2}}(u) \right]^2 \leq Z_\rho^{\nu_1}(u) Z_\rho^{\nu_2}(u).$$

Now, let choose $\nu_1 = \nu - 2$ and $\nu_2 = \nu$, then we obtain the Turán type inequality

$$f_\rho^\nu(u) = [Z_\rho^{\nu-1}(u)]^2 - Z_\rho^{\nu-2}(u) Z_\rho^\nu(u) \leq 0,$$

which is valid for all $\nu, \rho \in \mathbb{R}$ and $u > 0$. This in turn together with (2.7) implies that

$$\left[\frac{[Z_\rho^\nu(u)]'}{Z_\rho^\nu(u)} \right]' = - \left[\frac{Z_\rho^{\nu-1}(u)}{Z_\rho^\nu(u)} \right]' = - \frac{f_\rho^\nu(u)}{[Z_\rho^\nu(u)]^2} \geq 0,$$

i.e. the function $u \mapsto [Z_\rho^\nu(u)]' / Z_\rho^\nu(u)$ is increasing on $(0, \infty)$ for all $\nu, \rho \in \mathbb{R}$.

e. This follows also from part **c** of this theorem. More precisely, in view of (2.7) the Laguerre type inequality (2.2) is equivalent to the Turán type inequality

$$[Z_\rho^{\nu-n}(u)]^2 \leq Z_\rho^{\nu-n-1}(u) Z_\rho^{\nu-n+1}(u),$$

which clearly follows from (2.9) by choosing $\nu_1 = \nu - n - 1$ and $\nu_2 = \nu - n + 1$.

f. Let us recall the Chebyshev integral inequality [27, p. 40]: If $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable functions, both increasing or both decreasing and $p : [a, b] \rightarrow \mathbb{R}$ is a positive integrable function, then

$$(2.10) \quad \int_a^b p(t) f(t) dt \int_a^b p(t) g(t) dt \leq \int_a^b p(t) dt \int_a^b p(t) f(t) g(t) dt.$$

Note that if one of the functions f or g is decreasing and the other is increasing, then (2.10) is reversed. We shall use this inequality. For this by using (2.6) let us write $Z_\rho^{-\nu}(u)$ as

$$Z_\rho^{-\nu}(u) = \int_0^\infty e^{-ut} t^{\nu-1} e^{-t^{-\rho}} dt$$

and let $p(t) = e^{-ut}$, $f(t) = t^{\nu-1}$ and $g(t) = e^{-t^{-\rho}}$. Clearly f is increasing (decreasing) on $(0, \infty)$ if and only if $\nu \geq 1$ ($\nu \leq 1$). Since $g'(t)/g(t) = \rho t^{-\rho-1}$, it follows that g is increasing if and only if $\rho > 0$. Moreover,

$$\int_0^\infty p(t) dt = \int_0^\infty e^{-ut} dt = \frac{1}{u}$$

and

$$\int_0^\infty p(t) f(t) dt = \int_0^\infty e^{-ut} t^{\nu-1} dt = u^{-\nu} \int_0^\infty e^{-s} s^{\nu-1} ds = u^{-\nu} \Gamma(\nu).$$

Similarly, integration by parts and (2.6) imply

$$\int_0^\infty p(t) g(t) dt = \int_0^\infty e^{-ut} e^{-t^{-\rho}} dt = \frac{\rho}{u} \int_0^\infty t^{-\rho-1} e^{-t^{-\rho}} e^{-ut} dt = \frac{\rho}{u} Z_\rho^\rho(u).$$

Now, by choosing $\nu = 0$ in (2.1), and using (1.3) we obtain that

$$\rho Z_\rho^\rho(u) = u Z_1^{-1}(u) = 2\sqrt{u} K_1(2\sqrt{u})$$

and appealing to the Chebyshev integral inequality (2.10) the proof of the inequality (2.3) is done.

Finally, observe that by using the relation [32, p. 79] $K_\nu(u) = K_{-\nu}(u)$ we obtain easily

$$(2.11) \quad Z_1^{-\nu}(u) = u^{-\nu} Z_1^\nu(u),$$

and if we let $\rho = 1$ in (2.3), then by using (2.11) we immediately obtain (2.4), and with this the proof is complete. \square

3. Turán type inequalities for Krätzel functions

Let us consider now the Turán type inequality

$$(3.1) \quad [Z_\rho^\nu(u)]^2 - Z_\rho^{\nu-\rho}(u)Z_\rho^{\nu+\rho}(u) < 0,$$

which holds for all $\nu, \rho \in \mathbb{R}$ and $u > 0$. This inequality is actually a particular case of part **c** of Theorem 1. More precisely, by choosing $\nu_1 = \nu - \rho$ and $\nu_2 = \nu + \rho$ in (2.9) the proof of (3.1) is done. However, we give here an alternative proof. Just observe that

$$[Z_\rho^\nu(u)]^2 - Z_\rho^{\nu-\rho}(u)Z_\rho^{\nu+\rho}(u) = \frac{1}{2} \int_0^\infty \int_0^\infty (ts)^{\nu-1} e^{-t^\rho - s^\rho - u(\frac{1}{t} + \frac{1}{s})} [2 - (t/s)^\rho - (s/t)^\rho] dt ds$$

and by using the elementary inequality $(t/s)^\rho + (s/t)^\rho \geq 2$, the integrand becomes negative, which proves (3.1). Moreover the above integral representation yields the following complete monotonicity result: the function

$$u \mapsto \begin{vmatrix} Z_\rho^{\nu-\rho}(u) & Z_\rho^\nu(u) \\ Z_\rho^\nu(u) & Z_\rho^{\nu+\rho}(u) \end{vmatrix} = \frac{1}{2} \int_0^\infty \int_0^\infty (ts)^{\nu-1} e^{-t^\rho - s^\rho - u(\frac{1}{t} + \frac{1}{s})} [(t/s)^\rho + (s/t)^\rho - 2] dt ds$$

is not only positive, but even completely monotonic on $(0, \infty)$ for all $\nu, \rho \in \mathbb{R}$. The next result is an analogue of [19, Theorem 2.1] for the Turán determinant of Krätzel functions and provides a generalization of the above result and of part **b** of Theorem 1. Note that in view of (1.3) the following result in particular for $\rho = 1$ gives better Turán type inequalities for the modified Bessel function of the second kind K_ν than [19, Theorem 2.5]. For more details, compare the first Turán type inequality in [19, Remark 2.6] with the right-hand side of (3.3) below.

Theorem 2. *If $\nu, \rho \in \mathbb{R}$ and $n \in \{1, 2, \dots\}$, then the function*

$$u \mapsto A_{\rho,n}^\nu(u) = \begin{vmatrix} Z_\rho^{\nu-\rho}(u) & Z_\rho^\nu(u) & \dots & Z_\rho^{\nu+(n-1)\rho}(u) \\ Z_\rho^\nu(u) & Z_\rho^{\nu+\rho}(u) & \dots & Z_\rho^{\nu+n\rho}(u) \\ \vdots & \vdots & \ddots & \vdots \\ Z_\rho^{\nu+(n-1)\rho}(u) & Z_\rho^{\nu+n\rho}(u) & \dots & Z_\rho^{\nu+(2n-1)\rho}(u) \end{vmatrix}$$

is completely monotonic on $(0, \infty)$.

Proof. By using (1.1) we have

$$\begin{aligned} A_{\rho,n}^\nu(u) &= \begin{vmatrix} Z_\rho^{\nu-\rho}(u) & Z_\rho^\nu(u) & \dots & Z_\rho^{\nu+(n-1)\rho}(u) \\ Z_\rho^\nu(u) & Z_\rho^{\nu+\rho}(u) & \dots & Z_\rho^{\nu+n\rho}(u) \\ \vdots & \vdots & \ddots & \vdots \\ Z_\rho^{\nu+(n-1)\rho}(u) & Z_\rho^{\nu+n\rho}(u) & \dots & Z_\rho^{\nu+(2n-1)\rho}(u) \end{vmatrix} \\ &= \int_{[0,\infty)^{n+1}} \begin{vmatrix} t_0^{\nu-\rho-1} & t_0^{\nu-1} & \dots & t_0^{\nu+(n-1)\rho-1} \\ t_1^{\nu-1} & t_1^{\nu+\rho-1} & \dots & t_1^{\nu+n\rho-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_n^{\nu+(n-1)\rho-1} & t_n^{\nu+n\rho-1} & \dots & t_n^{\nu+(2n-1)\rho-1} \end{vmatrix} \prod_{j=0}^n e^{-t_j^\rho - \frac{u}{t_j}} dt_0 dt_1 \dots dt_n \\ &= \int_{[0,\infty)^{n+1}} \begin{vmatrix} t_{\sigma(0)}^{\nu-\rho-1} & t_{\sigma(0)}^{\nu-1} & \dots & t_{\sigma(0)}^{\nu+(n-1)\rho-1} \\ t_{\sigma(1)}^{\nu-1} & t_{\sigma(1)}^{\nu+\rho-1} & \dots & t_{\sigma(1)}^{\nu+n\rho-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{\sigma(n)}^{\nu+(n-1)\rho-1} & t_{\sigma(n)}^{\nu+n\rho-1} & \dots & t_{\sigma(n)}^{\nu+(2n-1)\rho-1} \end{vmatrix} \prod_{j=0}^n e^{-t_j^\rho - \frac{u}{t_j}} dt_0 dt_1 \dots dt_n, \end{aligned}$$

where σ is a permutation on $\{0, 1, \dots, n\}$. Now, let $\text{sgn}(\sigma)$ be denote the sign of σ and S_n be the symmetric group on n symbols. Then we obtain

$$\begin{aligned}
A_{\rho,n}^\nu(u) &= \int_{[0,\infty)^{n+1}} \begin{vmatrix} t_0^0 & t_0^\rho & \dots & t_0^{n\rho} \\ t_1^\rho & t_1^{2\rho} & \dots & t_1^{(n+1)\rho} \\ \vdots & \vdots & & \vdots \\ t_n^{n\rho} & t_n^{(n+1)\rho} & \dots & t_n^{2n\rho} \end{vmatrix} \text{sgn}(\sigma) \prod_{j=0}^n t_j^{\nu-\rho-1} e^{-t_j^\rho - \frac{u}{t_j}} dt_0 dt_1 \dots dt_n \\
&= \int_{[0,\infty)^{n+1}} \begin{vmatrix} 1 & t_0^\rho & \dots & t_0^{n\rho} \\ 1 & t_1^\rho & \dots & t_1^{n\rho} \\ \vdots & \vdots & & \vdots \\ 1 & t_n^\rho & \dots & t_n^{n\rho} \end{vmatrix} \text{sgn}(\sigma) (t_0^\rho t_1^\rho \dots t_n^{n\rho}) \prod_{j=0}^n \left(t_j^{\nu-\rho-1} e^{-t_j^\rho - \frac{u}{t_j}} \right) dt_0 dt_1 \dots dt_n \\
&= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \begin{vmatrix} 1 & t_0^\rho & \dots & t_0^{n\rho} \\ 1 & t_1^\rho & \dots & t_1^{n\rho} \\ \vdots & \vdots & & \vdots \\ 1 & t_n^\rho & \dots & t_n^{n\rho} \end{vmatrix} \prod_{j=0}^n \left(t_j^{\nu-\rho-1} e^{-t_j^\rho - \frac{u}{t_j}} \right) \\
&\quad \times \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) t_0^{\sigma(0)} (t_1^\rho)^{\sigma(1)} \dots (t_n^{n\rho})^{\sigma(n)} dt_0 dt_1 \dots dt_n \\
&= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \prod_{1 \leq i < j \leq n} (t_j^\rho - t_i^\rho) \prod_{j=0}^n \left(t_j^{\nu-\rho-1} e^{-t_j^\rho - \frac{u}{t_j}} \right) \\
&\quad \times \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) t_0^{\sigma(0)} (t_1^\rho)^{\sigma(1)} \dots (t_n^{n\rho})^{\sigma(n)} dt_0 dt_1 \dots dt_n \\
&= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \prod_{1 \leq i < j \leq n} (t_j^\rho - t_i^\rho)^2 \prod_{j=0}^n \left(t_j^{\nu-\rho-1} e^{-t_j^\rho - \frac{u}{t_j}} \right) dt_0 dt_1 \dots dt_n,
\end{aligned}$$

where we used that by Leibniz's formula the Vandermonde determinant can be written as

$$\begin{vmatrix} 1 & t_0^\rho & \dots & t_0^{n\rho} \\ 1 & t_1^\rho & \dots & t_1^{n\rho} \\ \vdots & \vdots & & \vdots \\ 1 & t_n^\rho & \dots & t_n^{n\rho} \end{vmatrix} = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) t_0^{\sigma(0)} (t_1^\rho)^{\sigma(1)} \dots (t_n^{n\rho})^{\sigma(n)} = \prod_{1 \leq i < j \leq n} (t_j^\rho - t_i^\rho).$$

Summarizing

$$A_{\rho,n}^\nu(u) = \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} e^{-\sum_{j=0}^n \left(t_j^\rho + \frac{u}{t_j} \right)} \prod_{1 \leq i < j \leq n} (t_j^\rho - t_i^\rho)^2 \prod_{j=0}^n t_j^{\nu-\rho-1} dt_0 dt_1 \dots dt_n$$

and then for all $\nu, \rho \in \mathbb{R}$, $u > 0$ and $m \in \{0, 1, \dots\}$ we obtain

$$\begin{aligned}
0 < (-1)^m [A_{\rho,n}^\nu(u)]^{(m)} &= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \left(\sum_{j=0}^n \frac{1}{t_j} \right)^m e^{-\sum_{j=0}^n \left(t_j^\rho + \frac{u}{t_j} \right)} \\
&\quad \times \prod_{1 \leq i < j \leq n} (t_j^\rho - t_i^\rho)^2 \prod_{j=0}^n t_j^{\nu-\rho-1} dt_0 dt_1 \dots dt_n,
\end{aligned}$$

which completes the proof. \square

Now, let us consider the function $\Phi_\rho^\nu : (0, \infty) \rightarrow \mathbb{R}$, defined by

$$\Phi_\rho^\nu(u) = 1 - \frac{Z_\rho^{\nu-\rho}(u) Z_\rho^{\nu+\rho}(u)}{[Z_\rho^\nu(u)]^2}.$$

Recall the asymptotic expansion (see [22, 24])

$$Z_\rho^\nu(u) \sim \alpha_\rho^\nu u^{\frac{2\nu-\rho}{2(\rho+1)}} e^{-\beta_\rho u^{\frac{\rho}{\rho+1}}},$$

where

$$\alpha_\rho^\nu = \sqrt{\frac{2\pi}{\rho+1}} \rho^{1-\frac{2\nu+1}{2(\rho+1)}} \quad \text{and} \quad \beta_\rho = (1+1/\rho) \rho^{\frac{1}{\rho+1}},$$

which holds for large values of u and fixed $\rho > 0$, $\nu \in \mathbb{R}$. By using the above asymptotic relation we obtain that $\lim_{u \rightarrow \infty} \Phi_\rho^\nu(u) = 0$, which shows that in (3.1) the constant 0 is the best possible. Moreover, based on numerical experiments we believe, but are unable to prove the following conjecture.

Conjecture. *If $\nu > \rho > 0$, then the function Φ_ρ^ν is strictly increasing on $(0, \infty)$, and consequently the following Turán type inequality holds*

$$(3.2) \quad \rho/(\rho - \nu) [Z_\rho^\nu(u)]^2 < [Z_\rho^\nu(u)]^2 - Z_\rho^{\nu-\rho}(u) Z_\rho^{\nu+\rho}(u).$$

Note that for $\nu, \rho > 0$ fixed if u tends to zero, then the asymptotic relation (see [22, 24])

$$\rho Z_\rho^\nu(u) \sim \Gamma(\nu/\rho)$$

is valid. Using this relation we obtain

$$\lim_{u \rightarrow 0} \Phi_\rho^\nu(u) = 1 - \frac{\Gamma(\nu/\rho - 1) \Gamma(\nu/\rho + 1)}{\Gamma^2(\nu/\rho)} = \rho/(\rho - \nu)$$

for all $\nu > \rho > 0$, which shows that in (3.2) the constant $\rho/(\rho - \nu)$ is the best possible.

It is worthwhile to note here that in fact the inequality (3.2) is motivated by the following result. If the above conjecture were true then (3.2) together with (3.1) would yield a generalization of (3.3), since for $\rho = 1$ the inequalities (3.2) and (3.1) reduce to (3.3).

Theorem 3. *Let K_ν be the modified Bessel function of the second kind. Then the following Turán type inequalities hold for all $\nu > 1$ and $u > 0$*

$$(3.3) \quad 1/(1 - \nu) [K_\nu(u)]^2 < [K_\nu(u)]^2 - K_{\nu-1}(u) K_{\nu+1}(u) < 0.$$

Moreover, the right-hand side of (3.3) holds true for all $\nu \in \mathbb{R}$. These inequalities are sharp in the sense that the constants $1/(1 - \nu)$ and 0 are the best possible.

For the sake of completeness it should be mentioned that the right-hand side of (3.3) was first proved by M.E.H. Ismail and M.E. Muldoon [20], and later by A. Laforgia and P. Natalini [26] and recently was deduced also by Á. Baricz [11, 12] and J. Segura [29], by using different approaches. The left-hand side of (3.3) was deduced very recently by using completely different methods by Á. Baricz [12] and J. Segura [29]. See also [13] for more details on (3.3). Note that the left-hand side of (3.3) provides actually an upper bound for the effective variance of the generalized Gaussian distribution. More precisely, in [1] the authors used (without proof) the inequality $0 < v_{\text{eff}} < 1/(\mu - 1)$ for $\mu = \nu + 4$, where

$$v_{\text{eff}} = \frac{K_{\mu-1}(u) K_{\mu+1}(u)}{[K_\mu(u)]^2} - 1$$

is the effective variance of the generalized Gaussian distribution.

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